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Dynamics of spinning charged particles in ambient electromagnetic fields

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1 General considerations

A classical spinning point particle on the spacetime $(\mathcal{M}, g)$ is modelled on an affinely-parametrized timelike future-directed curve C and a spacelike vector field S over C:

$$
\begin{aligned}
&C : I \subset \mathbb{R} \to \mathcal{M} \\
&\qquad \tau \mapsto p = C(\tau), \\
&\dot{C} = C_* \partial_\tau, \\
&g(\dot{C}, \dot{C}) = -1, \\
&S \in T_p\mathcal{M}\,,\ g(\dot{C}, S) = 0
\end{aligned}
$$

where I is an interval on $\mathbb{R}$, τ is the proper time of C and $p \in \mathcal{M}$ is any point in the image of C.

The metric tensor field g and the tangent vector field $\dot{C}$ over C are used to construct the spaces $T^{\|}_p\mathcal{M}$ and $T^{\perp}_p\mathcal{M}$ where

$$
\begin{aligned}
&T^{\|}_p\mathcal{M} = \{X \in T_p\mathcal{M}; X = f\dot{C}\,, f \in \mathbb{R}\}, \\
&T^{\perp}_p\mathcal{M} = \{X \in T_p\mathcal{M}; g(\dot{C}, X) = 0\}.
\end{aligned}
$$

Any vector field X over C has the unique decomposition $X = X_\| + X_\perp$ where $X_\| \in T^{\|}_p\mathcal{M}$ and $X_\perp \in T^{\perp}_p\mathcal{M}$. The $\dot{C}$-parallel and $\dot{C}$-orthogonal components of X are constructed using the projection maps $\Pi^{\|}_{\dot{C}}$ and $\Pi^{\perp}_{\dot{C}}$:

$$
\begin{aligned}
&\Pi^{\|}_{\dot{C}} : T_p\mathcal{M} \to T^{\|}_p\mathcal{M} \\
&\qquad X \mapsto \Pi^{\|}_{\dot{C}} X = -g(\dot{C}, X)\dot{C}
\end{aligned}
$$

and

$$\Pi^{\perp}_{\dot{C}} : T_p\mathcal{M} \to T^{\perp}_p\mathcal{M}$$
$$X \mapsto \Pi^{\perp}_{\dot{C}} X = X + g(\dot{C}, X)\dot{C}.$$

Let ∇ be the Levi-Civita connection on $(\mathcal{M}, g)$. The Fermi-Walker connection $\nabla^F_{\dot{C}}$ on vector fields over C is

$$\nabla^F_{\dot{C}} : T_p\mathcal{M} \to T_p\mathcal{M}$$
$$X \mapsto \nabla^F_{\dot{C}} X = \Pi^{\parallel}_{\dot{C}} \nabla_{\dot{C}} \Pi^{\parallel}_{\dot{C}} X + \Pi^{\perp}_{\dot{C}} \nabla_{\dot{C}} \Pi^{\perp}_{\dot{C}} X.$$

A Fermi-parallel vector field Y over C satisfies

$$\nabla^F_{\dot{C}} Y = 0.$$

By definition, the spin vector of an ideal gyroscope is Fermi-parallel.

The equations of motion for the pair (C, S) have the general form

$$\nabla_{\dot{C}} \dot{C} = \mathcal{A}[C, \dot{C}, S],$$
$$\nabla^F_{\dot{C}} S = \mathcal{T}[C, \dot{C}, S] \qquad (1)$$

where the acceleration $\mathcal{A}$ and torque $\mathcal{T}$ fields over C depend on $(C, \dot{C}, S)$. The vector fields $\mathcal{A}$ and $\mathcal{T}$ are induced from spacetime tensor fields and depend on further properties of the particle. Note that $\mathcal{T}$ and $\mathcal{A}$ cannot be chosen arbitrarily; by acting with $\Pi^{\parallel}_{\dot{C}}$ on (1) we find that

$$\Pi^{\parallel}_{\dot{C}} \mathcal{A} = 0,$$
$$\Pi^{\parallel}_{\dot{C}} \mathcal{T} = 0 \qquad (2)$$

i.e. both $\mathcal{A}$ and $\mathcal{T}$ are $\dot{C}$-orthogonal.

Using the Fermi-Walker connection and the projection maps associated with $\dot{C}$ one may motivate expressions for $\mathcal{A}$ and $\mathcal{T}$ using their non-relativistic Newtonian counterparts. An ad-hoc, but physically justifiable, rule is to replace temporal derivatives of Newtonian objects by Fermi-Walker derivatives of their spacetime counterparts. The equations obtained using this approach are equivalent to those obtained by specifying $\mathcal{A}$ and $\mathcal{T}$ in the instantaneous rest frame of the particle and transforming them to the lab frame.

2 The Thomas-Bargmann-Michel-Telegdi equation

The conventional description of a classical spinning point particle with mass m and charge q in an ambient electromagnetic field F is given by the Lorentz force law and the Thomas-Bargmann-Michel-Telegdi (TBMT) equation:

$$\nabla_{\dot{C}}\dot{C} = \frac{q}{m}\widetilde{\iota_{\dot{C}}F},$$
$$\nabla_{\dot{C}}S = \frac{q}{m}\left[\frac{\mathfrak{g}}{2}\widetilde{\iota_S F} + \frac{1}{2}(2-\mathfrak{g})\iota_S\iota_{\dot{C}}F\dot{C}\right] \tag{3}$$

where $\mathfrak{g}$ is the particle's gyromagnetic ratio and $\tilde{}$ is the metric isomorphism between $T\mathcal{M}$ and $T^*\mathcal{M}$.

The TBMT equation may be obtained starting from

$$\nabla^F_{\dot{C}}S = \frac{\mathfrak{g}q}{2m}\Pi^{\perp}_{\dot{C}}\widetilde{\iota_S F}. \tag{4}$$

The motivation behind (4) is that the spin vector S of a particle at rest on Minkowski spacetime (i.e. C is a timelike geodesic) in an ambient magnetic field behaves as expected.

Using $g(\dot{C}, S) = 0$ we find that

$$\begin{aligned}\nabla^F_{\dot{C}}S &= \Pi^{\perp}_{\dot{C}}\nabla_{\dot{C}}S \\ &= \nabla_{\dot{C}}S + g(\dot{C}, \nabla_{\dot{C}}S)\dot{C} \\ &= \nabla_{\dot{C}}S - g(\nabla_{\dot{C}}\dot{C}, S)\dot{C} \\ &= \nabla_{\dot{C}}S - \frac{q}{m}\iota_S\iota_{\dot{C}}F\dot{C}.\end{aligned}$$

and furthermore

$$\Pi^{\perp}_{\dot{C}}\widetilde{\iota_S F} = \widetilde{\iota_S F} + \iota_{\dot{C}}\iota_S F\dot{C}$$

which combined with (4) leads to the TBMT equation.

3 The Stern-Gerlach force

The Lorentz-TBMT system of equations (3) has been studied extensively in the accelerator literature. However, this model neglects the effects of the Stern-Gerlach forces which may be significant for the electron. Previous attempts to generalize the Lorentz-TBMT system encountered self-inconsistent sets of differential equations, which may be shown to stem from choices for $\mathcal{A}$ and $\mathcal{T}$ that do not satisfy (2). Our approach is to ensure that the choices we make *manifestly* satisfy (2).

In an instantaneous rest frame the conventional expression for the Stern-Gerlach force on a classical spinning point particle in an applied magnetic field **B** is proportional to grad(**S**·**B**) where **S** is the spin vector of the particle. A simple covariant expression for the Stern-Gerlach force that is manifestly orthogonal to $\dot{C}$ is

$$\mathcal{F}_{\rm SG} = \frac{\mathfrak{g}q}{2m}\big(\iota_S\iota_{\dot{C}} \star \nabla_{X_a}F\big)\Pi^{\perp}_{\dot{C}}X^a$$

where the set $\{X_a\}$ is a basis for $T\mathcal{M}$ with the dual basis $\{e^a\}$ for $T^*\mathcal{M}$:

$$e^a(X_b) = \delta^a_b$$

and

$$X^a = \tilde{e^a}.$$

The total force on the particle is the sum of the Lorentz force and the Stern-Gerlach force :

$$m\mathcal{A} = q\iota_{\dot{C}}F + \frac{\mathfrak{g}q}{2m}\big(\iota_S\iota_{\dot{C}} \star \nabla_{X_a}F\big)\Pi^{\perp}_{\dot{C}}X^a \tag{5}$$

and we adopt the same torque as in the previous section :

$$\mathcal{T} = \frac{\mathfrak{g}q}{2m}\Pi^{\perp}_{\dot{C}}\widetilde{\iota_S F}. \tag{6}$$

The dynamical consequences of (5) and (6) will be explored in later work.